

On the existence of a torsor structure for Galois covers over a complete discrete valuation ring

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Abstract

In this note we investigate the problem of existence of a torsor structure for Galois covers of (formal) schemes over a complete discrete valuation ring of residue characteristic $p > 0$ in the case of abelian Galois p -groups.

§0. Introduction

In this paper R denotes a *complete discrete valuation ring*, with uniformiser π , residue field k of characteristic $p > 0$, and fraction field $K := \text{Fr}R$. For an R -(formal)scheme Z we write $Z_K := Z \times_{\text{Spec}R} \text{Spec}K$ and $Z_k := Z \times_{\text{Spec}R} \text{Spec}k$ for the generic and special fibre, respectively, of Z . (In the case where Z is a formal R -scheme by its generic fibre Z_K we mean the associated rigid analytic space.) Let X be a (*formal*) R -scheme of finite type which is *normal*, geometrically connected, and flat over R . We further assume that the special fibre X_k of X is *integral*. Let $f_K : Y_K \rightarrow X_K$ be an *étale torsor* under a finite étale K -group scheme \tilde{G} of rank p^t ($t \geq 1$), with Y_K *geometrically connected*, and $f : Y \rightarrow X$ the corresponding morphism of *normalisation*. (Thus, Y is the normalisation of X in Y_K .) We are interested in the following question.

Question 1. *When is $f : Y \rightarrow X$ a torsor under a finite and flat R -group scheme G which extends \tilde{G} , i.e., with $G_K = \tilde{G}$?*

The following is well known.

Theorem A. (Proposition 2.4 in [Saïdi]; Theorem 5.1 in [Tossici]) *If $\text{char}(K) = 0$ we assume that R contains a primitive p -th root of 1, and X is locally factorial. Let η be the generic point of X_k and \mathcal{O}_η the local ring of X at η , which is a discrete valuation ring with fraction field $K(X)$: the function field of X . Let $f_K : Y_K \rightarrow X_K$ be an étale torsor under a finite étale K -group scheme \tilde{G} of **rank** \mathbf{p} , with Y_K connected, and let $K(X) \rightarrow L$ be the corresponding extension of function fields. Assume that the ramification index above \mathcal{O}_η in the field extension $K(X) \rightarrow L$ equals 1. Then $f : Y \rightarrow X$ is a torsor under a finite and flat R -group scheme G of rank p which extends \tilde{G} (i.e., with $G_K = \tilde{G}$).*

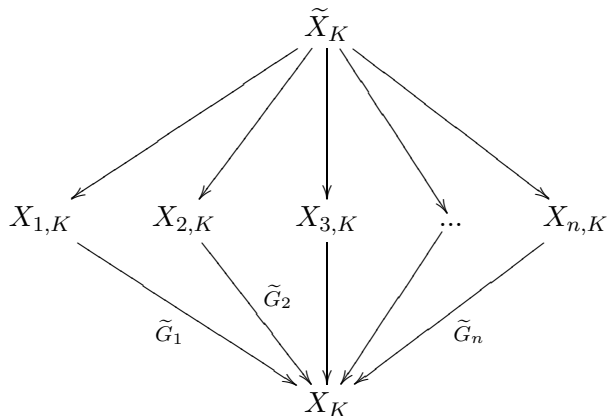
Strictly speaking the above references treat the case where $\text{char}(K) = 0$. For the equal characteristic $p > 0$ case see [Saïdi1], Theorem 2.2.1. Theorem A also holds when

Next, we describe the setting in this paper. Let $n \geq 1$, and for $i \in \{1, \dots, n\}$ let

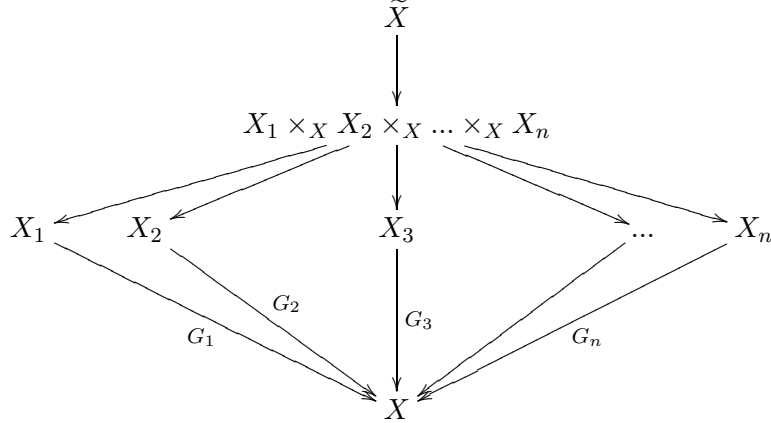
be an étale torsor under an étale finite *commutative* K -group scheme \tilde{G}_i , with $X_{i,K}$ *geometrically connected*, such that the $\{f_{i,K}\}_{i=1}^n$ are *generically pairwise disjoint*. Assume that $f_{i,K} : X_{i,K} \rightarrow X_K$ extends to a torsor

under a finite and flat (necessarily commutative) R -group scheme G_i with $(G_i)_K = \tilde{G}_i$, and with X_i normal, $\forall i \in \{1, \dots, n\}$. (Thus, X_i is the normalisation of X in $X_{i,K}$.) Let

and \tilde{X} the *normalisation* of X in \tilde{X}_K . Thus, \tilde{X}_K is the generic fibre of \tilde{X} and we have the following commutative diagrams



and



where $X_1 \times_X X_2 \times_X \dots \times_X X_n$ denotes the fibre product of the $\{X_i\}_{i=1}^n$ over X , the morphism $\tilde{X} \rightarrow X_1 \times_X X_2 \times_X \dots \times_X X_n$ is birational and is induced by the natural *finite* morphisms $\tilde{X} \rightarrow X_i$, $\forall i \in \{1, \dots, n\}$. Note that $f_K : \tilde{X}_K \rightarrow X_K$ (resp. $\tilde{f} : X_1 \times_X X_2 \times_X \dots \times_X X_n \rightarrow X$) is a torsor under the étale finite commutative K -group scheme $\tilde{G} := \tilde{G}_1 \times_{\text{Spec } K} \tilde{G}_2 \times_{\text{Spec } K} \dots \times_{\text{Spec } K} \tilde{G}_n$ (resp. a torsor under the finite and flat commutative R -group scheme $G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$), as follows easily from the various definitions. Note that $(G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n)_K = \tilde{G}$.

In this setup Question 1 reads as follows.

Question 2. *When is $f : \tilde{X} \rightarrow X$ a torsor under a finite and flat (necessarily commutative) R -group scheme G which extends \tilde{G} , i.e., with $G_K = \tilde{G}$?*

Our main result in this paper is the following.

Theorem B. *We use the same notations as above. Assume that \tilde{X}_k is **reduced**. Then the following three statements are equivalent.*

1. *$f : \tilde{X} \rightarrow X$ is a torsor under a finite and flat commutative R -group scheme G , in which case $G = G_1 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$ necessarily.*
2. *$\tilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$, in other words $X_1 \times_X X_2 \times_X \dots \times_X X_n$ is normal.*
3. *$(X_1 \times_X X_2 \times_X \dots \times_X X_n)_k$ is reduced.*

Note that the above condition in Theorem B that \tilde{X}_k is reduced is always satisfied after possibly passing to a finite extension R'/R of R (cf. [Epp]). It implies that the $(X_i)_k$ are reduced, $\forall i \in \{1, \dots, n\}$. Moreover, Theorem A and Theorem B provide a “complete” answer to Question 1 in the case of Galois covers of type (p, \dots, p) , i.e., the case where $\text{rank}(G_i) = p$, $\forall i \in \{1, \dots, n\}$.

In the case of (relative) *smooth curves* one can prove the following more precise result.

Theorem C. *We use the same notations and assumptions as in Theorem B. Assume further that X is a (relative) **smooth R -curve**, $n \geq 2$, and R is **strictly henselian**. If $\text{char}(K) = 0$ we assume that K contains a primitive p -th root of 1. Then the three (equivalent) conditions in Theorem B are equivalent to the following.*

4. **At least $n-1$** of the finite flat R -group schemes G_i acting on $f_i : X_i \rightarrow X$ are **étale**, for $i \in \{1, \dots, n\}$.

Remarks D. 1. *Theorem B holds true if X is the formal spectrum of a complete discrete valuation ring (cf. the details of the proof of Theorem B in §1 which applies as it is in this case).*

2. *In §3 we provide examples showing that Theorem C doesn't hold in relative dimension > 1 .*

§1. Proof of Theorem B

In this section we prove Theorem B. We start by the following.

Proposition 1.1 *Let G be a finite and flat commutative R -group scheme whose generic fibre is a product of group schemes of the form*

$$G_K = \tilde{G}_1 \times_{\text{Spec} K} \tilde{G}_2 \cdots \times_{\text{Spec} K} \tilde{G}_n,$$

where the $\{\tilde{G}_i\}_{i=1}^n$ are finite and flat commutative K -group schemes. Then G is a product of finite and flat commutative R -group schemes $\{G_i\}_{i=1}^n$, i.e.,

$$G = G_1 \times_{\text{Spec} R} G_2 \times_{\text{Spec} R} \cdots \times_{\text{Spec} R} G_n,$$

with $(G_i)_K = \tilde{G}_i$.

Proof. First, we treat the case $n = 2$. Thus, we have $G_K = \tilde{G}_1 \times_{\text{Spec} K} \tilde{G}_2$ and need to show $G = G_1 \times_{\text{Spec} R} G_2$ where $(G_i)_K = \tilde{G}_i$, for $i = 1, 2$. Let G_i be the *schematic closure* of \tilde{G}_i in G , for $i = 1, 2$ (cf. [Raynaud], 2.1). Therefore, G_1 and G_2 are closed subgroup schemes of G which are finite and flat over $\text{Spec} R$ (cf. loc. cit.). We have a short exact sequence

$$1 \rightarrow G_1 \rightarrow G \rightarrow G/G_1 \rightarrow 1,$$

and likewise

$$1 \rightarrow G_2 \rightarrow G \rightarrow G/G_2 \rightarrow 1,$$

of finite and flat commutative R -group schemes (cf. loc. cit.). It remains for the proof to show that the composite homomorphism $G_2 \rightarrow G \rightarrow G/G_1$ is an isomorphism. The morphism $G \rightarrow G/G_1$ is finite. The morphism $G_2 \rightarrow G$ is a closed immersion, hence finite. The composite $G_2 \rightarrow G/G_1$ of the above morphisms is then finite. We will show it is an isomorphism. The morphism $G_2 \rightarrow G/G_1$ is a closed immersion since its kernel is trivial. Indeed, on the generic fibre the kernel is trivial: $(G_1 \cap G_2)_K = \tilde{G}_1 \cap \tilde{G}_2 = \{1\}$. The map $G_2 \rightarrow G/G_1$ is then an isomorphism as both group schemes have the same rank.

Similarly, the morphism $G_1 \rightarrow G/G_2$ is an isomorphism. Therefore, $G = G_1 \times_{\text{Spec} R} G_2$ as required. Now an easy devissage argument along the above lines of thought, using induction on n , reduces immediately to the above case $n = 2$. \square

Proof of Theorem B

Proof. (1 \Rightarrow 2) Assume that $f : \tilde{X} \rightarrow X$ is a torsor under a finite and flat R -group scheme G . In particular, $G_K = \tilde{G}$ and G is necessarily commutative. We will show that $\tilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$, i.e., show that $X_1 \times_X X_2 \times_X \dots \times_X X_n$ is normal (this will imply that $G = G_1 \times_{\text{Spec} R} \dots \times_{\text{Spec} R} G_n$ necessarily, as $G_1 \times_{\text{Spec} R} \dots \times_{\text{Spec} R} G_n$ is the group scheme of the torsor $\tilde{f} : X_1 \times_X X_2 \times_X \dots \times_X X_n \rightarrow X$). One reduces easily by a devissage argument to the case $n = 2$ which we will treat below.

Assume $n = 2$. We have the following commutative diagrams of torsors

$$\begin{array}{ccc}
 & \tilde{X}_K & \\
 \tilde{G}_2 \swarrow & & \searrow \tilde{G}_1 \\
 X_{1,K} & & X_{2,K} \\
 \tilde{G}_1 \searrow & & \swarrow \tilde{G}_2 \\
 & X_K &
 \end{array}$$

and

$$\begin{array}{ccccc}
 & & \tilde{X} & & \\
 & \swarrow^{G'_2} & \downarrow & \searrow^{G'_1} & \\
 & X_1 \times_X X_2 & & & \\
 & \swarrow^{G_2} & & \searrow^{G_1} & \\
 X_1 & & & & X_2 \\
 & \searrow^{G_1} & & \swarrow^{G_2} & \\
 & X & & &
 \end{array}$$

where $\tilde{X} \rightarrow X_i$ is a torsor under a finite and flat R -group scheme G'_i , for $i = 1, 2$. Moreover, $G'_1 = \left(\tilde{G}_1 \right)^{\text{schematic closure}}$, and $G'_2 = \left(\tilde{G}_2 \right)^{\text{schematic closure}}$ (where the schematic closure is taken inside G) holds necessarily, so that $G = G'_1 \times_{\text{Spec} R} G'_2$ (cf. Proposition 1.1). Note that $\tilde{X}/G'_1 = X_2$ must hold as the quotient \tilde{X}/G'_1 is normal: since $\left(\tilde{X}/G'_1 \right)_k$ is reduced (as \tilde{X}_k is reduced and \tilde{X} dominates \tilde{X}/G'_1), and $\left(\tilde{X}/G'_1 \right)_K = X_{2,K}$ is normal (cf. [Liu], 4.1.18). Similarly $\tilde{X}/G'_2 = X_1$ holds. We want to show that $\tilde{X} = X_1 \times_X X_2$,

and we claim that this reduces to showing that the natural morphism $G \rightarrow G_1 \times_{\text{Spec } R} G_2$ (cf. the map ϕ below) is an isomorphism. Indeed, if one has two torsors, in this case $\tilde{X} \rightarrow X$ and $X_1 \times_X X_2 \rightarrow X$ above the same X , under isomorphic group schemes, which are isomorphic on the generic fibres, and if we have a morphism $\tilde{X} \rightarrow X_1 \times_X X_2$ which is compatible with the torsor structure and the given identification of group schemes (cf. above diagrams and the definition of ϕ below), then this morphism must be an isomorphism. (This is a consequence of Lemma 4.1.2 in [Tossici]. In [Tossici] $\text{char}(K) = 0$ is assumed, the same proof however applies if $\text{char}(K) = p$.) We have two short exact sequences of finite and flat commutative R -group schemes (cf. above diagrams and discussion for the equalities $G_1 = G/G'_2$ and $G_2 = G/G'_1$)

$$1 \rightarrow G'_2 \rightarrow G \rightarrow G_1 = G/G'_2 \rightarrow 1,$$

and

$$1 \rightarrow G'_1 \rightarrow G \rightarrow G_2 = G/G'_1 \rightarrow 1.$$

The morphisms $G \rightarrow G_1$, and $G \rightarrow G_2$, are finite. Consider the following exact sequence

$$1 \rightarrow \text{Ker}(\phi) \rightarrow G \rightarrow G_1 \times_{\text{Spec } R} G_2,$$

where $\phi : G \rightarrow G_1 \times_{\text{Spec } R} G_2$ is the morphism induced by the above morphisms. We want to show that the map $\phi : G \rightarrow G_1 \times_{\text{Spec } R} G_2$ is an isomorphism. We have $\text{Ker}(\phi) = G'_1 \cap G'_2$ by construction. However, $G'_1 \cap G'_2 = \{1\}$ since $G = G'_1 \times_{\text{Spec } R} G'_2$ by Proposition 1.1, and therefore $\text{Ker}(\phi) = \{1\}$ which means $\phi : G \rightarrow G_1 \times_{\text{Spec } R} G_2$ is a closed immersion. Finally, G and $G_1 \times_{\text{Spec } R} G_2$ have the same rank as group schemes which implies ϕ is an isomorphism, as required.

(2 \Rightarrow 3) Clear.

(3 \Rightarrow 1) By assumption $(X_1 \times_X X_2 \times_X \dots \times_X X_n)_k$ is reduced. Moreover, we have $(X_1 \times_X X_2 \times_X \dots \times_X X_n)_K = \tilde{X}_K$ is normal. Hence $X_1 \times_X X_2 \times_X \dots \times_X X_n$ is normal (cf. [Liu], 4.1.18), and $\tilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$. We know that $\tilde{f} : X_1 \times_X X_2 \times_X \dots \times_X X_n \rightarrow X$ is a torsor under the group scheme $G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$, so $f : \tilde{X} \rightarrow X$ is a torsor under the group scheme $G = G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$. \square

§2. Proof of Theorem C

In this section we prove Theorem C.

Proof. (1 \Rightarrow 4) Suppose that $\tilde{f} : \tilde{X} \rightarrow X$ is a torsor under a finite and flat R -group scheme G ; in which case $\tilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$ and $G = G_1 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$ (cf. Theorem B). We will show that *at least* $n - 1$ of the finite flat R -group schemes G_i (acting on $f_i : X_i \rightarrow X$) are étale, for $i \in \{1, \dots, n\}$. We argue by induction on the rank of G .

Base case: The base case pertains to $\text{rank}(G) = p^2$ and $n = 2$. Thus, $\text{rank}(G_1) = \text{rank}(G_2) = p$. We assume $\tilde{X} = X_1 \times_X X_2$ and prove that at least one of the two group

schemes G_1 or G_2 is étale. We assume that X is a scheme, and not a formal scheme, in which case the argument of proof is the same.

Let x be a *closed* point of X and \mathcal{X} the *boundary of the formal germ* of X at x , so \mathcal{X} is isomorphic to $\text{Spec}(R[[T]]\{T^{-1}\})$ (cf. [Saïdi2], §1). We have a natural morphism $\mathcal{X} \rightarrow X$ of schemes. Write $\mathcal{X}_1 := \mathcal{X} \times_X X_1$, $\mathcal{X}_2 := \mathcal{X} \times_X X_2$, and $\tilde{\mathcal{X}} := \mathcal{X} \times_X \tilde{X}$. Thus, by base change, $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ (resp. $\mathcal{X}_1 \rightarrow \mathcal{X}$, and $\mathcal{X}_2 \rightarrow \mathcal{X}$) is a torsor under the group scheme G (resp. under G_1 , and G_2) and we have the following commutative diagram

$$\begin{array}{ccc}
 & \tilde{\mathcal{X}} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2 & \\
 G_2 \swarrow & & \searrow G_1 \\
 \mathcal{X}_1 & & \mathcal{X}_2 \\
 G_1 \searrow & & \swarrow G_2 \\
 & \mathcal{X} &
 \end{array}$$

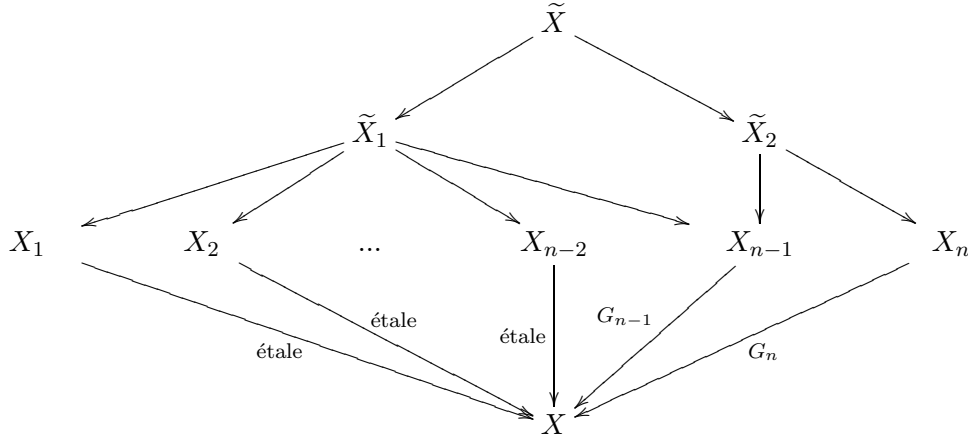
Note that $\tilde{\mathcal{X}}$ is normal as $(\tilde{\mathcal{X}})_k$ is reduced (recall $(\tilde{X})_k$ is reduced) and $(\tilde{\mathcal{X}})_K$ is normal (cf. [Liu], 4.1.18), hence $\tilde{\mathcal{X}} = \mathcal{X}_2 \times_{\mathcal{X}} \mathcal{X}_2$ holds (cf. Theorem B and Remarks D, 1).

Assume now that G_1 and G_2 are both *non-étale* R -group schemes. Then we prove that $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ can not have the structure of a torsor under a finite and flat R -group scheme which would then be a contradiction. More precisely, we will prove that $\mathcal{X}_2 \times_{\mathcal{X}} \mathcal{X}_2$ can not be normal in this case, hence the above conclusion (cf. Theorem B).

We will assume for simplicity that $\text{char}(K) = 0$ and K contains a primitive p -th root of 1. A similar argument used below holds in equal characteristic $p > 0$. First, \mathcal{X} is connected as \tilde{X}_k is *unibranch* (the finite morphism $\tilde{X}_k \rightarrow X_k$ is radicial). As the group schemes G_1 and G_2 are non étale, their special fibres $(G_1)_k$ and $(G_2)_k$ are radicial isomorphic to either μ_p or α_p . We treat the case $(G_1)_k$ is isomorphic to $\mu_p := \mu_{p,k}$ and $(G_2)_k$ is isomorphic to $\alpha_p := \alpha_{p,k}$, the remaining cases are treated similarly. (Recall \mathcal{X} is isomorphic to $\text{Spec}(R[[T]]\{T^{-1}\})$.) For a suitable choice of the parameter T the torsor $\mathcal{X}_2 \rightarrow \mathcal{X}$ is given by an equation $Z_2^p = 1 + \pi^{np}T^m$ where n is a positive integer (satisfying a certain condition) and $m \in \mathbb{Z}$ (cf. [Saïdi2], Proposition 2.3 (b). Strictly speaking in loc. cit. this is shown to hold after a finite extension of R , however a close inspection of the proof in loc. cit. reveals that this finite extension can be chosen to be étale. Also see Proposition 2.3.1 in [Saïdi3] for the equal characteristic case), and the torsor $\mathcal{X}_1 \rightarrow \mathcal{X}$ is given by an equation $Z_1^p = f(T)$ where $f(T) \in R[[T]]\{T^{-1}\}$ is a unit whose reduction $\overline{f(T)}$ modulo π is not a p -power (cf. loc. cit.). We claim that $\tilde{\mathcal{X}} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ can not hold. Indeed, by base change $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2 \rightarrow \mathcal{X}_2$ is a G_1 -torsor which is generically given by an equation $Z^p = f(T)$, where $f(T)$ is viewed as a function on \mathcal{X}_2 . But in \mathcal{X}_2 the function T becomes a p -power modulo π as one easily deduces from the equation $Z_2^p = 1 + \pi^{np}T^m$ defining the torsor $\mathcal{X}_2 \rightarrow \mathcal{X}$. In particular, the reduction $\overline{f(T)}$ modulo π of $f(T)$, viewed as a function on $(\mathcal{X}_2)_k$, is a p -power. This means that $(\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2)_k$ is not reduced and $\tilde{\mathcal{X}} \rightarrow \mathcal{X}_2$ can not be a $G_1 \simeq \mu_{p,R}$ -torsor (cf. the proof of Proposition 2.3 in [Saïdi2]), and a fortiori $\tilde{\mathcal{X}} \neq \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$.

Inductive hypothesis: Given G , we assume that the $(1 \Rightarrow 4)$ part in Theorem C holds true for $n \geq 2$ and cases where $\text{rank}(G_1) + \dots + \text{rank}(G_n) < \text{rank}(G)$. Write $\tilde{X}_1 := X_1 \times_X X_2 \times_X \dots \times_X X_{n-1}$. Then \tilde{X}_1 is normal (since its special fibre is reduced (as it is dominated by \tilde{X} whose special fibre is reduced) and its generic fibre is normal (cf. [Liu], 4.1.18)), hence at least $n-2$ of the corresponding G_i 's, for $i \in \{1, \dots, n-1\}$, are étale by the induction hypothesis. We will assume, without loss of generality, that G_i is étale for $1 \leq i \leq n-2$.

Inductive step: We have the following picture for our inductive step (the case for n):



We argue by contradiction. Suppose that neither G_{n-1} nor G_n is étale. This would mean that $\tilde{X}_2 \rightarrow X$, where \tilde{X}_2 is the normalisation of X in $(X_{n-1})_K \times_{X_K} (X_n)_K$, does not have the structure of a torsor (as this would contradict the induction hypothesis). This implies that $\tilde{X} \rightarrow X$ does not have the structure of a torsor since it factorises $\tilde{X} \rightarrow \tilde{X}_2 \rightarrow X$, for otherwise $\tilde{X}_2 \rightarrow X$ being a quotient of $\tilde{X} \rightarrow X$ would be a torsor. Of course, $\tilde{X} \rightarrow X$ is a torsor to start with by assumption and so this is a contradiction. Therefore, at least one of G_{n-1} and G_n is étale, as required.

$(1 \Leftarrow 4)$ Suppose that at least $n-1$ of the G_i are étale, say: G_1, G_2, \dots, G_{n-1} are étale. Write $\tilde{X}_1 := X_1 \times_X X_2 \times_X \dots \times_X X_{n-1}$. Then $\tilde{X}_1 \rightarrow X$ is a torsor under the finite étale R -group scheme $G'_1 := G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_{n-1}$. Moreover, $X_1 \times_X X_2 \times_X \dots \times_X X_n = \tilde{X}_1 \times_X X_n$, and $X_1 \times_X X_2 \times_X \dots \times_X X_n \rightarrow X_n$ is an étale torsor under the group scheme G'_1 (by base change). In particular, $(X_1 \times_X X_2 \times_X \dots \times_X X_n)_k$ is reduced as $(X_n)_k$ is reduced. (Indeed, \tilde{X} dominates X_n and \tilde{X}_k is reduced.) Hence $\tilde{X} = X_1 \times_X X_2 \times_X \dots \times_X X_n$ (cf. Theorem B) and $\tilde{X} \rightarrow X$ is a torsor under the group scheme $G := G_1 \times_{\text{Spec } R} G_2 \times_{\text{Spec } R} \dots \times_{\text{Spec } R} G_n$. \square

§3. Counterexample to Theorem C in higher dimensions

Theorem C is not valid (under similar assumptions) for (formal) smooth R -schemes of relative dimension ≥ 2 . Here is a counterexample. Assume $\text{char}(K) = 0$ and K contains

a primitive p -th root of 1. Let $X = \mathrm{Spf}(A)$ where $A := R \langle T_1, T_2 \rangle$ is the free R -Tate algebra in the two variables T_1 and T_2 . Let $G_1 = G_2 = \mu_p := \mu_{p,R}$, neither being an étale R -group scheme. For $i = 1, 2$, consider the G_i -torsor $X_i \rightarrow X$ which is generically defined by the equation

$$Z_i^p = T_i.$$

We have the following commutative diagram

$$\begin{array}{ccc}
 & X_1 \times X_2 & \\
 \mu_p \swarrow & & \searrow \mu_p \\
 & (Z'_2)^p = T_2 & (Z'_1)^p = T_1 \\
 X_1 & & X_2 \\
 \mu_p \searrow & & \swarrow \mu_p \\
 & X = \mathrm{Spf}(R \langle T_1, T_2 \rangle) & \\
 & Z_1^p = T_1 & Z_2^p = T_2
 \end{array}$$

The torsor $X_1 \times_X X_2 \rightarrow X_2$ is a $G_1 = \mu_p$ -torsor defined generically by the equation

$$(Z'_1)^p = T_1$$

where T_1 is viewed as a function on X_2 . This function is not a p -power modulo π as follows easily from the fact that the torsor $X_2 \rightarrow X$ is defined generically by the equation $Z_2^p = T_2$. In particular, $X_1 \times_X X_2 \rightarrow X_2$ is a non trivial μ_p -torsor, and $(X_1 \times_X X_2)_k \rightarrow (X_2)_k$ is a non trivial $\mu_{p,k}$ -torsor. Hence $(X_1 \times_X X_2)_k$ is necessarily reduced (as $(X_2)_k$ is reduced since $(X_2)_k \rightarrow (X_1)_k$ is a non trivial $\mu_{p,k}$ -torsor). Thus, $X_1 \times_X X_2$ is normal (cf. Theorem B) and $X_1 \times_X X_2 = \tilde{X}$, where \tilde{X} is the normalisation of X in $(X_1 \times_X X_2)_K$, which contradicts the statement of Theorem C in this case.

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